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Energy stability of the Ekman boundary layer

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The critical value R_E of the Reynolds number R is predicted by the application of the energy theory. When $R < R_E$, the Ekman layer is the unique steady solution of the Navier–Stokes equations and the same boundary conditions, and is, further, stable in a slightly weaker sense than asymptotically stable in the mean. The critical value R_E is determined by numerically integrating the relevant Euler–Lagrange equations. An analytic lower bound to R_E is obtained. Comparisons are made between R_E and R_L, the critical value of R according to linear theory, in order to demark the region of parameter space, $R_E < R < R_L$, in which subcritical instabilities are allowable.

1. Introduction

The Ekman boundary layer can exist for fluids in a rotating system (Greenspan 1968). It arises for a flow of a fluid in the semi-infinite space beneath a rigid surface which is moving over it with a constant velocity. Similarly, it also arises in the case of a fluid in a semi-infinite space moving over a stationary rigid surface where the fluid velocity far from the surface is uni-directional and constant.

The stability of the Ekman layer has been studied both experimentally (Faller 1963; Faller & Kaylor 1966; Tatro & Mollö-Christensen 1967) and through linearized stability theory (Barcilon 1965; Faller & Kaylor 1966; Lilly 1966). Combined, these studies confirm the existence of two classes of unstable disturbances. Class A instabilities, caused by the interaction of shear and Coriolis forces, are travelling waves. Class B waves, an inflexional instability, are nearly stationary with respect to the rotating plate. A more detailed discussion can be found in Greenspan (1968). The class A waves appear at a lower Reynolds number than those of class B. Lilly (1966) finds that $R_L = 55$ at an orientation $16°$ to the right of the motion of the plate with a preferred non-dimensional wave-number of 0·32.

We here investigate the energy stability of the Ekman layer. We seek a value R_E of R such that $R < R_E$ is a sufficient condition for stability in a slightly weaker sense than asymptotic stability in the mean (see § 4). Furthermore, if $R < R_E$, the Ekman layer is the unique steady solution of the Navier–Stokes equations subject to the same boundary conditions over the class of functions considered. An analytic lower bound to R_E is obtained.

The effect of disturbance amplitudes on the onset of instability is demarked. Regardless of initial amplitude, the critical value of the Reynolds number R_C for the onset of instability must lie in the interval $R_E < R_C < R_L$.

2. The basic flow field

The basic flow field is an exact solution (Barcilon 1965) to the incompressible Navier–Stokes equations. The system represented by this field consists of fluid of constant density ρ_0, and kinematic viscosity ν occupying the half-space below an infinite horizontal rigid plane. The entire fluid plane system rotates in a counter clockwise direction about a vertical axis with constant angular velocity $\mathbf{\Omega}$. In a Cartesian co-ordinate system, rotating with angular velocity $\mathbf{\Omega}$, z is taken to be the vertical co-ordinate with its positive direction downward, and x and y are taken in the plane of the surface. The velocity vector is $\mathbf{V}$, $\mathbf{V} = (u, v, w)$, the pressure is p, and the angular velocity is $\mathbf{\Omega} = \Omega(0, 0, -1)$ with respect to a non-rotating system; the surface at $z = 0$ is made to move in the y-direction with constant speed V_0 while the fluid as $z \to \infty$ remains at rest.

The system described satisfies the equations of motion in a rotating co-ordinate system,

$$\left.\begin{aligned} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V} + 2\mathbf{\Omega}\times\mathbf{V} &= -\nabla(p/\rho_0) + \nu\nabla^2\mathbf{V}, \\ \nabla.\mathbf{V} &= 0, \end{aligned}\right\} \tag{2.1a,b}$$

and the boundary conditions,

$$\left.\begin{array}{lll} \mathbf{V} = (0, V_0, 0) & \text{on} & z = 0, \\ \mathbf{V} \to 0 & \text{as} & z \to \infty. \end{array}\right\} \tag{2.1c,d}$$

The relevant solution of the system (2.1) is called the non-divergent Ekman boundary layer, and was given by Barcilon (1965) as follows:

$$\left.\begin{aligned} W &\equiv 0, \\ U &= -V_0 e^{-z/L}\sin(z/L), \\ V &= V_0 e^{-z/L}\cos(z/L), \end{aligned}\right\} \tag{2.2a,b,c}$$

where $L = (\nu/\Omega)^{\frac{1}{2}}$ and $\mathbf{V} = (U, V, W)$.

We shall study the stability of the solution represented by (2.2). The stability analysis will also apply to the above problem where instead of the plate moving uni-directionally and the fluid at $z \to \infty$ at rest, the fluid at $z \to \infty$ moves uni-directionally with constant speed V_0 while the fluid at the plate is at rest with respect to the rotating plate. The basic solution differs from (2.2) only trivially.

3. The energy identity

We will now develop the energy identity for disturbances to the basic state (2.2). Let $(\mathbf{V}, p)$ represent the velocity and pressure fields of the basic state, and let $(\mathbf{V}^*, p^*)$ represent any other solution of system (2.1). Now let

$$\left.\begin{aligned} \mathbf{u} = (u, v, w) &= \mathbf{V}^* - \mathbf{V}, \\ \pi &= p^* - p. \end{aligned}\right\} \tag{3.1a,b}$$

$(\mathbf{u}, \pi)$ represents the difference between the disturbed and undisturbed states. Since both $(\mathbf{V}, p)$ and $(\mathbf{V}^*, p^*)$ satisfy system (2.1), we can obtain the system governing the disturbance:

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} + \mathbf{u}.\nabla\mathbf{V} + \mathbf{V}.\nabla\mathbf{u} &= -\nabla(\pi/\rho_0) + \nu\nabla^2\mathbf{u}, \\ \nabla.\mathbf{u} &= 0, \\ \mathbf{u} = 0 \quad \text{on} \quad z &= 0, \\ \mathbf{u} \to 0 \quad \text{as} \quad z &\to \infty. \end{aligned}\right\} \tag{3.2a–d}$$

We shall henceforth confine our attention to disturbance functions which belong to the class $\mathscr{F}$:

$$\left.\begin{aligned} \mathscr{F} = \Big\{ f \Big| \int_0^\infty |f|\, dz < \infty, f \text{ is either} \\ &\text{(i) periodic in } x \text{ and } y, \\ &\text{(ii) Fourier transformable in } x \text{ and } y, \text{ or} \\ &\text{(iii) periodic in either } x \text{ or } y \text{ and Fourier transformable in the other.} \end{aligned}\right\} \tag{3.3}$$

The development of the energy identity now follows, with trivial changes in notation, that of Dudis & Davis (1971). The identity is as follows:

$$\frac{dK}{dt} = -\int_{\mathscr{V}} [R\mathbf{u}.\mathbf{D}.\mathbf{u} + \nabla\mathbf{u}:\nabla\mathbf{u}], \tag{3.4a}$$

where the Reynolds number R is defined by

$$R = \frac{V_0 L}{\nu} = V_0(\nu\Omega)^{-\frac{1}{2}} \tag{3.4b}$$

and

$$K = \int_{\mathscr{V}} \tfrac{1}{2}\mathbf{u}.\mathbf{u}, \tag{3.4c}$$

$$\mathbf{D} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & U_z \\ 0 & 0 & V_z \\ U_z & V_z & 0 \end{bmatrix}, \tag{3.4d}$$

and the scaled basic state is given by

$$(U, V) = (-e^{-z}\sin z,\ e^{-z}\cos z). \tag{3.4e}$$

The subscript z denotes differentiation, and we have non-dimensionalized the identity using the following scales: length $\sim L$, time $\sim L^2/\nu$, velocity $\sim V_0$ where L and V_0 are defined below (2.2). We have not distinguished non-dimensional variables from dimensional ones, since we shall henceforth confine our attention to only non-dimensional quantities.

4. The maximum problem

Let us define

$$D = \int_{\mathscr{V}} \nabla\mathbf{u} : \nabla\mathbf{u} \tag{4.1a}$$

and

$$I = -\int_{\mathscr{V}} \mathbf{u}.\mathbf{D}.\mathbf{u}. \tag{4.1b}$$

With these definitions, the energy identity (3.4a) can be written

$$\frac{dK}{dt} = -\left(1 - R\frac{I}{D}\right) D. \tag{4.2}$$

It follows from (4.2) that

$$\frac{dK}{dt} \leqslant -\left(1 - \frac{R}{R_E}\right) D, \tag{4.3}$$

whenever the maximum problem (M) is satisfied:

$$R_E^{-1} = \max_{\mathscr{S}} I, \quad D = 1 \tag{M}$$

where

$\mathscr{S} = \{\mathbf{u} | \mathbf{u}$ has continuous second partial derivatives, $\nabla.\mathbf{u} = 0$, $\mathbf{u} = 0$ on $z = 0$ and $u, v, w \in \mathscr{F}\}$.

Precisely the same arguments made in § 4 of Dudis & Davis (1971) apply here. We state the result:

If $R < R_E$, then the basic state is stable in that $D \rightarrow 0$ as $t \rightarrow \infty$ in the sense that

$$\lim_{\tau\rightarrow\infty} \int_0^\tau D\,dt < \infty. \tag{4.4}$$

Let $\tilde{\mathscr{V}}$ be a rectangular parallelepiped bounded in the z-direction with the wall $z = 0$ as one of its boundaries. The extent in the x- and y-directions is either a wavelength or the whole real line depending on which case of class $\mathscr{F}$ is being considered. Then, if there exists a positive number $\tilde{a}^2$ such that

$$\int_{\tilde{\mathscr{V}}} \nabla\mathbf{u} : \nabla\mathbf{u} \geqslant \tilde{a}^2 \int_{\tilde{\mathscr{V}}} \mathbf{u}.\mathbf{u}$$

holds, then the definition of stability (4.4) further means that $\tilde{K} \equiv \int_{\tilde{\mathscr{V}}} \tfrac{1}{2}\mathbf{u}.\mathbf{u} \rightarrow 0$ as $\tau \rightarrow \infty$ in the sense that

$$\lim_{\tau\rightarrow\infty} \int_0^\tau \tilde{K}\,dt < \infty. \tag{4.5}$$

Furthermore, if $R < R_E$, then the Ekman layer represented by equations (2.2) is the unique steady solution of the Navier–Stokes equations and boundary conditions (2.1c, d) over the class $\mathscr{F}$.

The maximum problem (M) is equivalent to the variational equation:

$$\delta\left\{\int_{\mathscr{V}} \mathbf{u}.\mathbf{D}.\mathbf{u} - \frac{2}{R}\int_{\mathscr{V}} p\nabla.\mathbf{u} + \frac{1}{R}\int_{\mathscr{V}} \nabla\mathbf{u} : \nabla\mathbf{u}\right\} = 0,$$

where p and R are Lagrange multipliers. The consequent Euler–Lagrange equations are as follows:

$$\left.\begin{aligned} R\mathbf{u}.\mathbf{D} &= -\nabla p + \nabla^2\mathbf{u}, \\ \nabla.\mathbf{u} &= 0, \end{aligned}\right\} \qquad (4.6a, b)$$

with the boundary conditions

$$\left.\begin{aligned} \mathbf{u} &= 0 \quad \text{on} \quad z = 0, \\ \mathbf{u} &\to 0 \quad \text{as} \quad z \to \infty. \end{aligned}\right\} \qquad (4.6c, d)$$

As in Dudis & Davis (1971) it is easy to show that

$$R_E = \min \operatorname{pos} R,$$

and we have stability and uniqueness in the above sense if

$$R < R_E.$$

5. The eigenvalue R_E

5.1. *Lower bounds*

Lower bounds to R_E can be obtained as in Dudis & Davis (1971) by considering the maximum principle (M) over a space of functions $\mathscr{S}_1$ extended from $\mathscr{S}$ by relaxing the restriction $\nabla.\mathbf{u} = 0$. As was obtained there, the consequent Euler–Lagrange equations corresponding to the lower bound are as follows:

$$\left.\begin{aligned} \tfrac{1}{2}\bar{R}U_z w &= \nabla^2 u, \\ \tfrac{1}{2}\bar{R}V_z w &= \nabla^2 v, \\ \tfrac{1}{2}\bar{R}U_z u + \tfrac{1}{2}\bar{R}V_z v &= \nabla^2 w, \end{aligned}\right\} \qquad (5.1a\text{–}c)$$

with the conditions

$$\left.\begin{aligned} \mathbf{u} &= 0 \quad \text{on} \quad z = 0. \\ \mathbf{u} &\to 0 \quad \text{as} \quad z \to \infty. \end{aligned}\right\} \qquad (5.1d, e)$$

Here $R_w = \inf \operatorname{pos} \bar{R}$ and $R_w \leqslant R_E$. Again, an analytic lower bound R_{ww} to R_w can be found by using

$$U_z \leqslant \sqrt{2}\, e^{-z}$$

and

$$V_z \leqslant \sqrt{2}\, e^{-z}.$$

The system (5.1) is identical to the system (7.3) of Dudis & Davis (1971) for $P = 1$ if we make the transformation,

$$(x, y, z, u, v, w) \to (-y, z, x, -w, \phi, u).$$

Hence we find that, using the results of Dudis & Davis (1971, § 7),

$$R_w = 3{\cdot}09,$$

$$R_{ww} = 1{\cdot}45,$$

and that $R_{ww} \leqslant R_w \leqslant R_E$.

5.2. *Reduction to ordinary differential equations*

Let us now express those functions of $\mathscr{F}$, case (i) in terms of normal modes as follows:

$$f = \mathrm{Re}\{\hat{f}(l,k,z)\exp[i(lx+ky)]\}.$$

In case (ii), we can write

$$f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(l,k,z)\exp[i(lx+ky)]\,dl\,dk.$$

The Euler–Lagrange system (5.1) in these cases as well as in case (iii), becomes the following:

$$\left.\begin{aligned} \tfrac{1}{2}RU_z\hat{w} &= -il\hat{p}+(D^2-m^2)\,\hat{u},\\ \tfrac{1}{2}RV_z\hat{w} &= -ik\hat{p}+(D^2-m^2)\,\hat{v},\\ \tfrac{1}{2}RU_z\hat{u}+\tfrac{1}{2}RV_z\hat{v} &= -D\hat{p}+(D^2-m^2)\,\hat{w},\\ il\hat{u}+ik\hat{v}+D\hat{w} &= 0, \end{aligned}\right\} \tag{5.2}$$

with the boundary conditions,

$$\hat{u} = \hat{v} = \hat{w} = 0 \quad \text{on} \quad z = 0,$$

$$\hat{u}, \hat{v}, \hat{w} \to 0 \quad \text{as} \quad z \to \infty,$$

where

$$D = d/dz \quad \text{and} \quad m = (k^2+l^2)^{\frac{1}{2}}.$$

5.3. *Relationship with the buoyancy boundary layer*

The Euler–Lagrange system (5.2) can be compared with the corresponding system, (8.1) of Dudis & Davis (1971), governing the buoyancy boundary layer. The buoyancy problem for $P = 1$ and $k, \hat{v} \equiv 0$ is identical with the Ekman problem for $k = 0$. This can be seen by making the transformation

$$(x,y,z,k,l,\hat{u},\hat{v},\hat{w}) \to (-y,0,x,0,-l,\hat{w},\hat{\phi},\hat{u})$$

in system (5.2) and comparing with Dudis & Davis (1971, equation (8.1)). This result also holds in the linear stability problem (Gill & Davey 1969). No relation exists between the complete three-dimensional problems. All of the above are consequences of the analogies between rotating and stratified fluids (Veronis 1967).

5.4. *Symmetry property*

The smallest positive eigenvalue $\tilde{R}(k,l)$ of R of system (5.2) satisfies the symmetry relation,

$$\tilde{R}(k,l) = \tilde{R}(-k,-l). \tag{5.3}$$

Relation (5.3) can be obtained by transforming system (5.2) as follows:

$$(l,k,\hat{p},\hat{u},\hat{v},\hat{w},\tilde{R}) \to (-\bar{l},-\bar{k},-\bar{p},-\bar{u},-\bar{w},-\bar{v},\bar{R})$$

and noting that $\tilde{R} = \bar{R}$. Thus, due to (5.3), when seeking $R_E = \min\limits_{k,l}\tilde{R}(k,l)$, it is sufficient to search only a half-plane of the (k,l)-plane.

5.5. *Numerical solution*

The numerical scheme is identical to that of Dudis & Davis (1971, § 8.4). The appendix summarizes the details.

The $\min_{l} \tilde{R}(0, l)$ is the two-dimensional result that is identical to that of the two-dimensional buoyancy problem for $P = 1$ (see § 5.3). The result is

$$\min_{l} \tilde{R}(0, l) = 21{\cdot}3 \pm 0{\cdot}1.$$

The minimum is attained at $l = 0{\cdot}70 \pm 0{\cdot}05$. The search in the remainder of the half-plane yields that

$$R_E = 18{\cdot}3 \pm 0{\cdot}1,$$

which is attained for $k = 0{\cdot}41 \pm 0{\cdot}01$, $l = 0{\cdot}38 \pm 0{\cdot}01$. This is equivalently a wave-number of $m = 0{\cdot}56 \pm 0{\cdot}01$; the dependent variables are independent of an angle $47{\cdot}2° \pm 1{\cdot}5°$ to the right of the surface flow on $z = 0$. The results are entered in table 1.

	Wave-numbers	Angle
$R_L = 55$	0·32	16°
$R_E = 18{\cdot}3$	0·56	47·2°
$R_w = 3{\cdot}09$	0	—
$R_{ww} = 1{\cdot}45$	0	—

TABLE 1

6. Results and conclusions

The object of the numerical analysis was to find the value R_E. When $R < R_E$, the Ekman boundary layer is asymptotically stable in the mean over rectangular parallelepipeds bounded by the wall at $z = 0$ and of finite extent in the z-direction (see § 4), and is additionally the unique steady solution of the governing equations over the class $\mathscr{F}$.

Easily obtained lower bounds to R_E were developed by solving the maximum principle (M) over a space of functions not restricted by $\nabla . \mathbf{u} = 0$. The lower bound R_{ww} obtained analytically was $R_{ww} = 1{\cdot}45$ while a better bound R_w was obtained numerically as $R_w = 3{\cdot}09$ with an accuracy of $\pm 0{\cdot}01$. In both cases, these values correspond to the limit $m \to 0$. Also see table 1.

The energy limit R_E was computed numerically, and found to be $R_E = 18{\cdot}3$ with an accuracy of $\pm 0{\cdot}1$. The corresponding wave-numbers are

$$(k, l) = (0{\cdot}41, 0{\cdot}38)$$

with an accuracy of $\pm 0{\cdot}01$. This corresponds to eigenfunctions independent of a direction at an angle of $47{\cdot}2° \pm 1{\cdot}5°$ to the right of the surface flow at $z = 0$. See table 1.

The energy results are to be compared with the results of linear theory. Lilly (1966) finds the critical value R_L of the Reynolds number for instability as

$R_L = 55$ for a wave-number of 0·32. The eigenfunctions are independent of direction at an angle of 16° to the right of the flow at $z = 0$. See table 1. We thus have that subcritical instabilities may exist in the region $R_E < R < R_L$.

The results of experiments seem generally to be concentrated on the onset of the class B waves, since these are nearly stationary in the rotating system and are thus more easily detected. At this point there seems to be no record of observations of subcritical instabilities for class A waves (i.e. for $R < R_L$). Although this paper shows that subcritical instabilities are allowable, it does not show that they in fact exist. However, in all previous problems treated by the energy theory, when subcritical instabilities are allowable they do exist, so their existence is plausible here.

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Appendix. Numerical scheme

Let us define $\hat{a} = (D-m)\hat{u}, \quad \hat{b} = (D-m)\hat{v}$

and $u_j = u_{Rj} + iu_{Ij} \quad (j = 1, \ldots, 6).$

The Euler–Lagrange equations (5.2) can be written

$$\left.\begin{aligned}
D\hat{u}_R &= \hat{a}_R + m\hat{u}_R,\\
D\hat{u}_I &= \hat{a}_I + m\hat{u}_I,\\
D\hat{v}_R &= \hat{b}_R + \hat{v}_R,\\
D\hat{v}_I &= \hat{b}_I + m\hat{v}_I,\\
D\hat{w}_R &= l\hat{u}_I + k\hat{v}_I,\\
D\hat{w}_I &= -l\hat{u}_R - k\hat{v}_R,\\
D\hat{a}_R &= -l\hat{p}_I - m\hat{a}_R + \tfrac{1}{2}RU_z\hat{w}_R,\\
D\hat{a}_I &= l\hat{p}_R - m\hat{a}_I + \tfrac{1}{2}RU_z\hat{w}_I,\\
D\hat{b}_R &= -k\hat{p}_I - m\hat{b}_R + \tfrac{1}{2}RV_z\hat{w}_R,\\
D\hat{b}_I &= k\hat{p}_R - m\hat{b}_I + \tfrac{1}{2}RV_z\hat{w}_I,\\
D\hat{p}_R &= -\tfrac{1}{2}RU_z\hat{u}_R - \tfrac{1}{2}RV_z\hat{v}_R + l\hat{a}_I + k\hat{b}_I + km\hat{v}_I + lm\hat{u}_I - m^2\hat{w}_R,\\
D\hat{p}_I &= -\tfrac{1}{2}RU_z\hat{u}_I - \tfrac{1}{2}RV_z\hat{v}_I - l\hat{a}_R - k\hat{b}_R - km\hat{v}_R - lm\hat{u}_R - m^2\hat{w}_I.
\end{aligned}\right\} \quad \text{(A 1)}$$

As in appendix C of Dudis & Davis (1971) for large z, the asymptotic boundary conditions are the following:

$$M^3\hat{u} = M^3\hat{v} = M^3\hat{w} = 0 \quad \text{on} \quad x = x_1,$$

where $M = D + m$, and $x_1 \to \infty$ in fact but is taken to apply at finite values of x. $x_1 = 8$ was found sufficient to guarantee minor changes in the fifth significant figure of R_E for $x_1 > 8$. The system (A 1) was integrated to find R_E precisely, as in Dudis & Davis (1971).

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